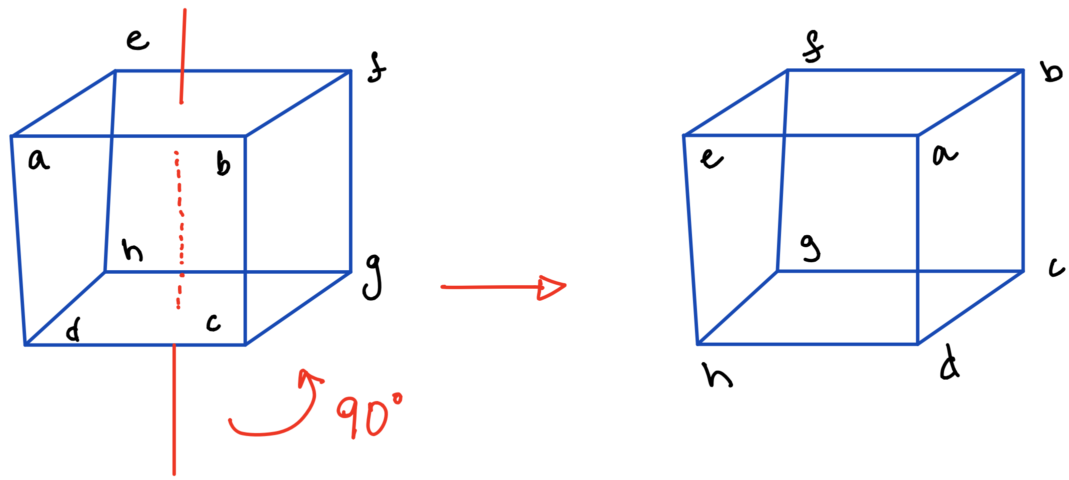
Lecture 24

We'll prove the orbit-stabilizer theorem in this lecture. Let's see why "should" the theorem be true. Below, we'll find the number of rotational symmetries of a cube in two different ways. We'll see later that both of them are actually versions of the orbit-stabilizer theorem.

Rotational Symmetries of a cube

Recall from our discussion of the dihedral group that a symmetry of an n-gon is a transformation which might change the places of vertices and edges but doesn't change the shape of the n-gon.

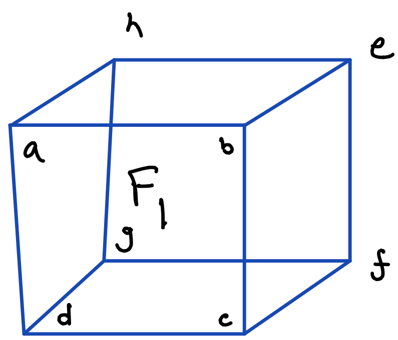
But we can do the same thing with a cube. Let G be the group of rotational symmetries of a cube.



so for example, the above figure demonstrates the rotation of the cube by $90°$ in the counterclockwise direction along the axis shown in red. The position of the vertices changed but the shape and size of the cube didn't. So this is an example of a rotational symmetry of the cube. We want to find out the # of all such symmetries, i.e., $|G|$.

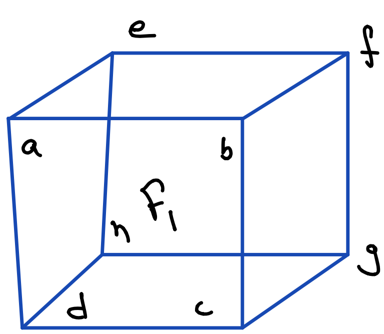
Of course, you can just find it by brute force. However, we will be smarter (or atleast, pretend to be) and calculate it combinatorially (which we'll see to be basically using the Orbit-Stabilizer theorem).

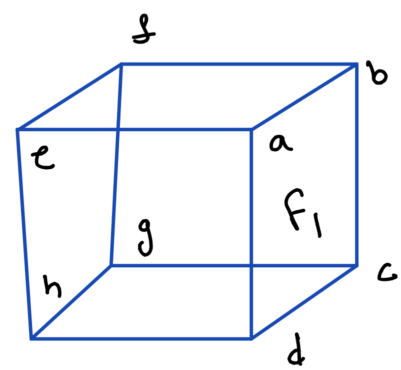
Method 1

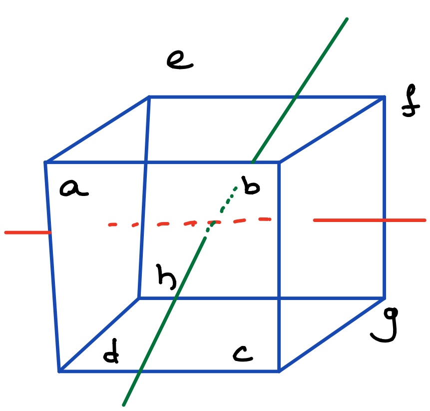


Consider the face F_1 of the cube (containing the vertices a, b, c, d). If we perform any rotational symmetry of the cube, the face F_1 might change its position. Since there are 6 faces in a cube so the # of places where F_1 can co $= 6$.

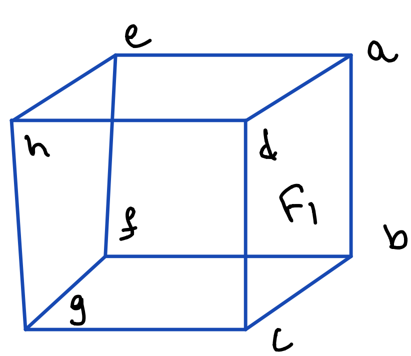
Now since we cannot change the shape and size of the cube, the vertices a, b, c, d of the face F_1 only have the liberty to move among themselves. So once F_1 has chosen its place, there are a total of 4 rotations which will keep F_1 fixed but permute the vertices among themselves. e.g.



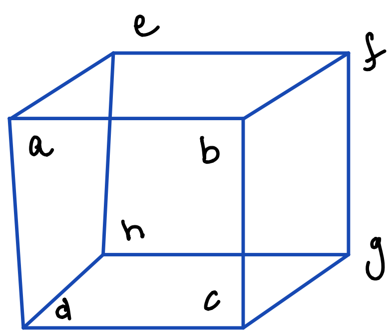


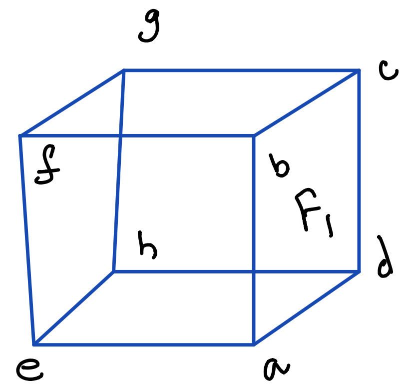


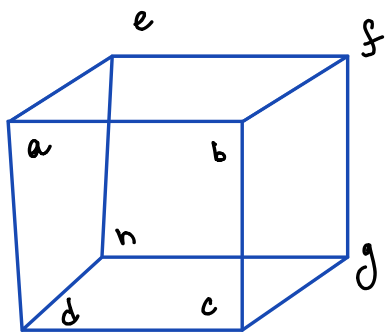
do 90° rotation around the red axis then 90° rotation about the green axis

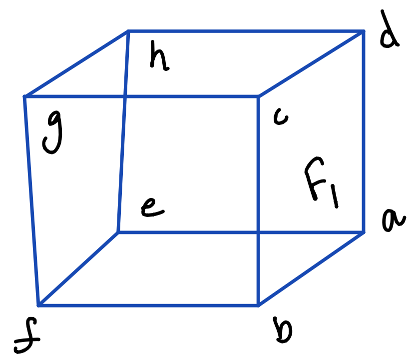


and similarly









In all the above figures, the face F_1 is at the same place, only the position of its vertices are changing.

But any rotational symmetry of the cube will do the same thing. So there are a total of $6 \times 4 = 24$ rotational symmetries.

$\Rightarrow \quad |G| = 24.$

Method 2

In method 1 we worked with a face. Here, let's work with a vertex. Suppose, we choose a vertex 'a' in the cube. Following any rotation, 'a' has 8 choices to move around. But since the rotation must be a symmetry $\Rightarrow$ the immediate neighbour vertices of 'a', 'b', 'd' and 'e' must be attached to it and can only move among themselves. So once 'a' has chosen a position, its immediate neighbours have 3 choices and hence for a rotational symmetry, there are $8 \times 3 = 24$ choices. So again

$$|G| = 24.$$

Now how does the Orbit-Stabilizer theorem relates to finding $|G|$?

Let G = group of rotational symmetries of the cube

F = set of faces of the cube

and consider the action of G on F by taking a rotational symmetry and a face, say F_1, apply the rotational symmetry to the cube and look at place which F_1 takes, which is again going to be some face of the cube and hence lies in F.

What is $|O_{F_1}|$? This is just the path which F_1 takes under the action by G but it is just the # of choices for F_1 = 6.

What is $|Stab(F_1)|$? Well, F_1 will be stabilized under the action if the rotation doesn't change the position of F_1. But it can still change the position of vertices of F_1 and hence $|Stab(F_1)| = 4$.

Since the O-S theorem says $|G| = |O_{F_1}||Stab(F_1)|$

$\Rightarrow |G| = 6 \cdot 4 = 24$.

So method 1 is just apply the orbit stabilizer theorem to a particular action of G.

But we can do the same thing in method 2! Take G as it is and now consider the set as V = set of vertices of the cube. The action of G on V is just pick a vertex, say 'a', act it by the rotational symmetry and look at it's new position which will again be in V. One can find (by a similar reasoning as above) that $|O_a| = 8$ and $|Stab(a)| = 3 \Rightarrow |G| = |O_a| \cdot |Stab(a)| = 24$.

So now that we have seen some applications of the O-S Theorem, let's now prove it.

Theorem [Orbit-Stabilizer Theorem]

Let G be a group which acts on a set X. Let $x \in X$. Then $[G : Stab(x)] = |O_x|$. If G is finite then $|O_x| = \frac{|G|}{|Stab(x)|} \Rightarrow |G| = |O_x||Stab(x)|$.

Proof Consider the set $C = \{ g\,Stab(x) \mid g \in G \}$, the set of all left cosets of $Stab(x)$ in G. $O_x = \{ g \cdot x \mid g \in G \}$. Define a map

$$T : C \longrightarrow O_x \quad \text{by}$$

$$T(g\,Stab(x)) = g \cdot x$$

We must check that T is well-defined (recall Principle 2) as it is map from the set of cosets.

<u>T is well-defined</u>

Let $g\,Stab(x) = h\,Stab(x) \Rightarrow h^{-1}g\,Stab(x) = Stab(x) \Rightarrow h^{-1}g \in Stab(x)$

$\Rightarrow (h^{-1}g)\cdot x = x$

$\Rightarrow h^{-1}\cdot(g\cdot x) = x$ [from point 2) ie the definition of a group action]

We can act by $h \in G$ on both sides of the above equation.

$h\cdot(h^{-1}\cdot(g\cdot x)) = h\cdot x \Rightarrow e\cdot(g\cdot x) = h\cdot x$

$\Rightarrow g\cdot x = h\cdot x$

$\Rightarrow T(g\,Stab(x)) = T(h\,Stab(x))$

and hence T is well-defined.

<u>T is one-one</u>

Let $T(g\,Stab(x)) = T(h\,Stab(x))$

$\Rightarrow \quad g \cdot x = h \cdot x$

$\Rightarrow \quad h^{-1} \cdot (g \cdot x) = h^{-1} \cdot (h \cdot x) = e \cdot x$

$\Rightarrow \quad (h^{-1}g) \cdot x = x \Rightarrow h^{-1}g \in Stab(x)$

(by the definition of $Stab(x)$)

$\Rightarrow \quad h^{-1}g\,Stab(x) = Stab(x)$

$\Rightarrow \quad g\,Stab(x) = h\,Stab(x)$

and hence T is one-one.

<u>T is onto</u>

Let $y \in O_x \Rightarrow \exists\, g \in G$ s.t. $y = g \cdot x$

Consider the coset $g\,Stab(x) \in C$. Then by the definition of T

$$T(g\,Stab(x)) = g \cdot x \Rightarrow T \text{ is onto.}$$

So T is a bijection b/w C and O_x.

But $|C| = [G : Stab(x)]$

$\Rightarrow \quad [G : Stab(x)] = |O_x|$

If G is finite $\Rightarrow [G : Stab(x)] = \dfrac{|G|}{|Stab(x)|}$

$\Rightarrow \quad |G| = |Stab(x)| \cdot |O_x|$

$\blacksquare$

Remark Note that the O-S Theorem holds for any group G with any action on any set X.

So for example, if G acts on itself by conjugation the we saw in Lec. 23 that $Stab(g) = C(g)$, the centralizer of g in G.

Thus in that case

$$[G : C(g)] = |O_g|.$$

As an application of the O-S Theorem, let's reprove Lagrange's Theorem.

<u>Lagrange's Theorem</u> If G is finite and $H \leq G$

$\Rightarrow$ $|H| \mid |G|$.

<u>Proof</u> Let $C = \{gH \mid g \in G\}$ be the set of left cosets of H in G. Consider the action of G on C by

$$G \times C \longrightarrow C$$
$$x, gH \longrightarrow xgH$$

i.e. multiply the group elements x and g and consider the coset containing xg.

Consider the element $H \in C$.

$$\begin{aligned} \mathrm{Stab}(H) &= \{x \in G \mid x \cdot H = H\} \\ &= \{x \in G \mid xH = H \Leftrightarrow x \in H\} \\ &= H \end{aligned}$$

So under this action $Stab(H) = H$.

But from the O-S theorem, as G is finite

$$|G| = |O_H||Stab(H)| \Rightarrow |Stab(H)| \, \big| \, |G|$$

$$\Rightarrow |H| \, \big| \, |G|$$

$\boxed{\text{lq}}$

So as you can see, we'll choose our set X as per our need and the action will be chosen accordingly. Then we can use the O-S Theorem to prove powerful theorems.

o ———— x ———— x ———— o